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An Extremal Problem of the Markov–Bernstein Type for Lacunary Polynomials

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INTRODUCTION AND NOTATION

The general extremal problem can be described as follows: For given fixed complex numbers a, b and z, $|z| = 1$, and for a given non-negative real function $f(z)$ defined on the unit circle, it is required to obtain an upper bound for $|ap_n(z) + bzp_n'(z)|$ where $p_n(z)$ varies over the class of all polynomials of degree not exceeding n which satisfy $|p_n(z)| \leqslant f(z)$ on the circle $|z| = 1$ [1, 3].

The case $f(z) = |P_\ell(z)|$, where $P_\ell(z)$ is a polynomial of degree $\ell \leqslant n$ is dealt with in detail in [3]. It is easy to reduce then the problem to the case $f(z) = |P_n(z)|$ where $P_n(z)$ is a polynomial of degree n, all of whose zeros lie in the closed unit disk.

In the present work we indicate the possibility of strengthening these results for lacunary polynomials. The precision is made by either diminishing the region of location of zeros of the polynomials concerned or by increasing the region of the validity of the relevant inequalities as compared with the case where no symmetry of the zeros of $P_n(z)$ is assumed. In all cases the results are sharp and include the existing theorems as special cases.

For $\rho \geqslant 1$ we shall denote by $D_{\rho,n}$ the image of the disk $|\zeta| < \rho$ by the function $w(\zeta) = n\zeta/(\zeta - 1)$, that is

$$D_{\rho,n} = \begin{cases} \left\{w \;\middle|\; \left| w - \dfrac{n\rho^2}{\rho^2 - 1} \right| > \dfrac{n\rho}{\rho^2 - 1}\right\} & \text{if } \rho > 1 \\ \left\{w \;\middle|\; \operatorname{Re} w < \dfrac{n}{2}\right\} & \text{if } \rho = 1 \end{cases}$$

Similarly, for z_0, $|z_0| \geqslant 1$ and $0 < r \leqslant 1$, we denote by $B^{(r)}_{|z_0|}$ the image of the disk $|\zeta| < r$ by the function $w_1(\zeta) = z_0/(z_0 - \zeta)$. For simplicity we set $B^{(1)}_{|z_0|} = B_{|z_0|}$, that is

$$B^{(r)}_{|z_0|} = \left\{w \;\middle|\; \left| w - \frac{|z_0|^2}{|z_0|^2 - r^2} \right| < \frac{r\,|z_0|}{|z_0|^2 - r^2}\right\} \qquad \text{if } 0 < r < 1$$

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and

$$B_{|z_0|} = \begin{cases} w \mid \left| w - \dfrac{|z_0|^2}{|z_0|^2 - 1} \right| < \dfrac{|z_0|}{|z_0|^2 - 1} & \text{if} \quad |z_0| > 1 \\ w \mid \operatorname{Re} w > \frac{1}{2} & \text{if} \quad |z_0| = 1 \end{cases}$$

One notices that $B_{|z_0|}^{(1/|z_0|)} = B_{|z_0|^2}$ and that the regions $B_{|z_0|}^{(r)}$, $B_{|z_0|}$ decrease monotonically with increasing $|z_0|$. The complement of a set S with respect to the extended complex plane will be denoted by $C(S)$ and the boundary of S will be denoted by ∂S.

Finally for $0 < \sigma \leqslant 1$ we let $D_{\sigma,n}^* = -D_{\rho,n} + n$ where $\rho = 1/\sigma$, that is,

$$D_{\sigma,n}^* = \begin{cases} \left\{ w \mid \left| w + \dfrac{n\sigma^2}{1 - \sigma^2} \right| > \dfrac{n\sigma}{1 - \sigma^2} \right\} & \text{if} \quad \sigma < 1 \\ \left\{ w \mid \operatorname{Re} w > \dfrac{n}{2} \right\} & \text{if} \quad \sigma = 1 \end{cases}$$

$D_{\sigma,n}^*$ can also be described as the image of the disk $|\zeta| < 1/\sigma$ by the function $w(\zeta) = n/(1 - \zeta)$.

THE MAIN LEMMA. *Let $Q_n(z)$ be a polynomial of degree $n (n \geqslant 1)$ whose zeros $z_k (k = 1, 2,..., n)$ all lie in the closed unit disk $|z| \leqslant 1$ and such that $\sum_{k=1}^{n} z_k = 0$.*

If

$$L_w\{Q_n(z)\} = zQ_n'(z) - w(z)\, Q_n(z)$$

where $w(z) \in \bar{D}_{\rho,n}$ for $|z| \geqslant \rho^{1/2}$, $(\rho \geqslant 1)$ then all the zeros of $L_w\{Q_n(z)\}$ lie in the disk $|z| \leqslant \rho^{1/2}$.

For $w =$ const. the result is best possible with the polynomial $Q_n(z) = C(z^2 - 1)^{n/2}$ for even n being extremal for the values $w = n\rho/(e^{i\theta} + \rho)$ $(0 \leqslant \theta < 2\pi)$. In this case $L_w\{Q_n(z)\}$ has at least the two zeros $\pm i\rho^{1/2}$ on the circumference $|z| = \rho^{1/2}$ and at the same time w traverses the boundary of $D_{\rho,n}$.

Proof. Obviously $L_w\{Q_n(z)\}$ vanishes at a double zero of $Q_n(z)$ and $L_w\{Q_n(z)\}$ does not vanish at a single zero of $Q_n(z)$. Assume $Q_n(z_0) \neq 0$. We have

$$\frac{L_w\{Q_n(z_0)\}}{nQ_n(z_0)} = \frac{1}{n} \sum_{k=1}^{n} w_k - \frac{w(z_0)}{n} = w^* - \frac{w(z_0)}{n}$$

where $w_k = z_0/(z_0 - z_k)$ and $w^* = (1/n) \sum_{k=1}^{n} w_k$. By a result proved in [2], $w^* = z_0/[z_0 - \alpha(z_0)]$ where $|\alpha(z_0)| \leqslant 1/|z_0|$ for $|z_0| > 1$. It follows therefore that if $L_w\{Q_n(z_0)\} = 0$ and $|z_0|^2 > \rho$, then $w^* = w(z_0)/n$. Since $w^* \in \bar{B}_{|z_0|}^{(1/|z_0|)} = B_{|z_0|^2}$ and $w(z_0)/n \in \bar{D}_{\rho,1} = C(B_\rho)$. It follows that

$$\bar{B}_{|z_0|^2} \cap C(B_\rho) \neq \varnothing \tag{1}$$

a contradiction to the relation $\bar{B}_{|z_0|^2} \subset B_\rho$.

This proves the first part of the lemma. The second part can be verified directly.

Remark 1. One can isolate the cases when a zero z_0 of $L_w\{Q_n(z)\}$ is on the circle $|z| = \rho^{1/2}$.

We distinguish two possibilities:

(a) $\rho > 1$. This implies

$$\frac{w}{n} = w^* \in \bar{B}_{|z_0|^2} \cap C(B_{|z_0|^2}) = \partial(\bar{B}_{|z_0|^2})$$

However by hypothesis

$$w^* = \frac{1}{z_0 - \alpha(z_0)} \in \partial \bar{B}_{|z_0|}^{1/|z_0|}$$

Therefore $|\alpha(z_0)| = 1/|z_0|$. By Schwarz's lemma $\alpha(z) = e^{i\vartheta}/z$. Hence $Q_n'(z)/Q_n(z) = nz/(z^2 - 1)$ and $Q_n(z) = C \cdot (z^2 - 1)^{n/2}$. The extremal polynomials are of even degree. The value of w can be calculated from the equation $L_w\{Q_n(z_0)\} = 0$. $w = nz_0^2/(z_0^2 - 1)$ traverses the circle $|w - n\rho^2/(\rho^2 - 1)| = n\rho/(\rho^2 - 1)$ as z_0^2 runs through the circle $|z| = \rho^{1/2}$.

(b) $|z_0| = \rho = 1$. Here we can have the case of a double zero of $Q_n(z)$. If however $Q_n(z_0) \neq 0$ then the conditions Re $w \leqslant n/2$, Re $w_k \geqslant n/2$, and $w^* = (1/n) \sum w_k$ imply that Re $w = n/2$, Re $w_k = \frac{1}{2}$. That is all the z_k lie on the unit circle. The extremal case is similar to (a).

Remark 2. One verifies easily from the proof of the lemma that if we assume that all the zeros of $Q_n(z)$ be in the disk $\{z \mid |z| < 1\}$ then all the zeros of $L_w\{Q_n(z)\}$ lie in the disk $\{z \mid |z| < \rho^{1/2}\}$.

Remark 3. As mentioned the lemma is sharp only for even degree polynomials $Q_n(z)$. It will be interesting to investigate the case of odd degree polynomials.

We mention briefly two corollaries.

COROLLARY 1. *If* $L_w\{Q_n(z_0)\} = 0$, $Q_n(z_0) \neq 0$ *for some* z_0 *with* $|z_0| \geqslant 1$ *then*

$$\frac{z_0 Q_n'(z_0)}{Q_n(z_0)} \in C(D_{|z_0|^2, n}).$$

Proof. This follows by the proof of the lemma from the relation $w^* \in \bar{B}_{|z_0|^2}$.

COROLLARY 2. (Generalization of Szegö's theorem). *Applying the main lemma with* $w(z) = nz/(z - \zeta)$ *we have*:

Let $Q_n(z)$ *be a polynomial of degree* $n(n \geqslant 1)$ *with zeros* z_k *with* $|z_k| \leqslant 1$, $\sum_{k=1}^{n} z_k = 0$. *Let* z_0 *and* ζ *be two numers such that* $|z_0| \geqslant 1$ *and* $|\zeta| \geqslant 1/|z_0|$. *Then*

$$(\zeta - z_0)\, Q_n'(z_0) + nQ_n(z_0) = 0$$

only if

(a) $\zeta = e^{i\theta}/z_0$ *and* $Q_n(z) = C(z^2 - e^{i\theta})^{n/2}$.

(b) $z_0(|z_0| = 1)$ *is double root of* $Q_n(z)$.

(c) $z_0 = \zeta(|z_0| = |\zeta| = 1)$ *and* $Q_n(z_0) = 0$.

THE MAIN RESULTS

THEOREM 1. *Let* $P_n(z)$ *be a polynomial of degree* $n(n \geqslant 1)$ *whose zeros* z_k *all lie in the disk* $|z| \leqslant 1$ *and let* $\sum_{k=1}^{n} z_k = 0$.

Let $\mathscr{P}_n$ *be the family of all polynomials* $p_n(z) = \sum_{k=1}^{n} a_k z^k$ *with* $a_{n-1} = 0$, *that satisfy* $|p_n(z)| \leqslant |P_n(z)|$ *on* $|z| = 1$.

Let $\rho(\rho \geqslant 1)$ *be fixed and let a and b be arbitrary complex numbers such that either* $b = 0$ *or* $a/b \in \bar{D}_{\rho,n}$.

Then for $|z| \geqslant \rho^{1/2}$

$$|bzp_n'(z) - ap_n(z)| \leqslant |bzP_n'(z) - aP_n(z)| \tag{2}$$

for all $p_n \in \mathscr{P}_n$.

Proof. Since the function $p_n(z)/P_n(z)$ is regular in the region $|z| \geqslant 1$ and since $|p_n(z)/P_n(z)| \leqslant 1$ on $|z| = 1$ it follows that either $p_n(z)/P_n(z) \equiv$ const. or $|p_n(z)| < |P_n(z)|$ for $|z| > 1$. The first case is trivial so we consider only the second case. Let $\zeta(|\zeta| < 1)$ be fixed. The zeros of $Q_n(z) = \zeta p_n(z) + P_n(z)$ are all in $|z| < 1$ and their sum vanishes. Hence by the main lemma applied to the polynomial $Q_n(z)$ and $w = a/b$ we have

$$z[\zeta p_n'(z) + P_n'(z)] - \frac{a}{b}[\zeta p_n(z) + P_n(z)] \neq 0 \qquad \text{for} \quad |z| \geqslant \rho^{1/2}$$

and $|\zeta| < 1$. Therefore $|\zeta|\ |bzp_n'(z) - ap_n(z)| < |bzP_n'(z) - aP_n(z)|$. Letting $|\zeta| \to 1$ we obtain the desired result.

COROLLARY 3. *In particular case when* $a = n \cdot \rho$, $b = \rho \pm 1$ $(\rho > 1)$ $P_n(z) = Mz^n (M > 0)$ *we obtain the inequality*

$$|(\rho \pm 1)\, zp'_n(z) - n\rho p_n(z)| \leqslant Mn\, |z|^n$$

valid for $|z| \geqslant \rho^{1/2}$ *and for* $p_n \in \mathscr{P}_n$ *which satisfy* $|p_n(z)| \leqslant M$ *on* $|z| = 1$. *For* $\rho \downarrow 1$ *we obtain the well known inequalities* $|2zp'_n(z) - np_n(z)| \leqslant Mn\,|z|^n$ *for* $|z| \geqslant 1$ *(See* [3] *p.* 393)

THEOREM 2. *Let* $P_n(z)$ *be a polynomial of degree* $n (n \geqslant 1)$ *whose zeros* z_k *all lie in the disk* $|z| \leqslant 1$ *and let* $\sum_{k=1}^{n} z_k = 0$.

Let $p_n(z) = \sum_{k=1}^{n} a_k z^k$ *be an arbitrary polynomial of degree not exceeding* n *such that* $a_{n-1} = 0$. *If* $\bar{S}$ *is the image of the region* $|z| \geqslant 1$ *by the funcion* $p_n(z)/P_n(z)$ *and if the numbers a and b are such that either* $b = 0$ *or* $b = 0$ *and* $a/b \in \bar{D}_{\rho,n}$, *then for all* z, $|z| \geqslant \rho^{1/2}$ $(\rho \geqslant 1)$

$$\frac{L\{p_n(z)\}}{L\{P_n(z)\}} = \frac{ap_n(z) - bzp'_n(z)}{aP_n(z) - bzP'_n(z)} \in \bar{S}$$

Proof. Let $\bar{K}$ be any closed circular region disjoint from $\bar{S}$ and let the function $\zeta = (\alpha w + \beta)/(\gamma w + \delta)$ map K onto the disk $|\zeta| < 1$. It follows that

$$\left| \frac{\alpha w + \beta}{\gamma w + \delta} \right| > 1 \qquad \text{for} \quad w \in \bar{S}.$$

Therefore

$$\left| \frac{\alpha[p_n(z)/P_n(z)] + \beta}{\gamma[p_n(z)/P_n(z)] + \delta} \right| > 1 \qquad \text{for} \quad |z| \geqslant 1.$$

Or

$$|\gamma p_n(z) + \delta P_n(z)| < |\alpha p_n(z) - \beta P_n(z)|$$

for all $|z| \geqslant 1$. This implies that all the zeros of $\alpha p_n(z) + \beta P_n(z)$ lie in the disk $|z| < 1$ and their sum vanishes. By Theorem 1

$$|\gamma L\{p_n(z)\} + \delta L\{P_n(z)\}| < |\alpha L\{p_n(z)\} + \beta L\{P_n(z)\}|$$

for $|z| \geqslant \rho^{1/2}$.

From this we conclude that

$$\frac{L\{p_n(z)\}}{L\{P_n(z)\}} \notin \bar{K} \qquad \text{for} \quad |z| \geqslant \rho^{1/2}.$$

Since K is an arbitrary circular region disjoint from $\bar{S}$, the result follows.

Remark 4. Theorem 1 applied to polynomials $z^n P_\ell(1/z)$ and $z^n p_m(1/z)$ where $P_\ell(z) = \sum_{k=0}^{\ell} a_k z^k$, $a_1 = 0$ is a polynomial which does not vanish in $|z| \leqslant 1$ and $p_m(z) = \sum_{k=0}^{m} b_k z^k$, $b_1 = 0$ $m \leqslant n$ and $\ell \leqslant n$ yields the relation (2) for $|z| \leqslant \sigma^{1/2}$ $(0 < \sigma \leqslant 1)$ provided $|p_m(z)| \leqslant |P_\ell(z)|$ on $|z| = 1$ and either $b = 0$ or $a/b \in \overline{D^*_{\sigma,n}}$.

COROLLARY 4. *For given* $M > 0$, $|\zeta| = 1$, $0 < \sigma \leqslant 1$ *we have* $|(1/\sigma^{1/2} - z)\, p_n'(z) + np_n(z)| \leqslant Mn$ *for* $|z| < \sigma^{1/2}$ *and for every polynomial* $p_n(z) = \sum_{k=0}^{n} b_k z^k$ *with* $b_1 = 0$ *satisfying* $|p_n(z)| \leqslant M$ *on* $|z| = 1$.

Proof. For $a = n$, $b = 1 - \zeta/\sigma^{1/2}z$ we have $a/b = n/1 - \zeta/z\sigma^{1/2}$ and for $|z| = \sigma^{1/2}$ a/b belongs to the image of the circle $|\zeta| = 1/\sigma$ by the function $w(\zeta) = n/(1 - \zeta)$, that is on the boundary of $D^*_{\sigma,n}$. Hence, letting $a = n$, $b = 1 - \zeta/\sigma^{1/2}z$, $P_n(z) = M$ we obtain from (2)

$$\left|\left(\frac{\zeta}{\sigma^{1/2}} - z\right) p_n'(z) + np_n(z)\right| \leqslant Mn \qquad \text{for} \quad |z| \leqslant \sigma^{1/2}.$$

Hence

$$\frac{1}{\sigma^{1/2}}\, |p_n'(z)| + |np_n(z) - zp_n'(z)| \leqslant Mn.$$

Remark. For $\sigma \to 0$ we have $|b_0| + 2|b_2|/n \leqslant M$.

For every polynomial $p_n(z) = \sum_{k=0}^{n} b_k z^k$ satisfying $|p_n(z)| \leqslant M$ on $|z| = 1$ we have $|b_0| + \frac{1}{2}|b_2| \leqslant M$. To prove the last relation let $K(\theta) = 1 + \frac{1}{2}(e^{2i\theta} - e^{-2i\theta})$. Then $(1/2\pi)\int_0^{2\pi} K(\theta)\, d\theta = 1$ and $\|p_n(e^{i\theta}) * K\|_\infty \leqslant M$ which yields $|b_0| + \frac{1}{2}|b_2| \leqslant M$.

We may generalize our main results for arbitrary lacunary polynomials. To this end the main lemma will be replaced by the following:

LEMMA. *Let* $Q_n(z) = z^n + a_{n-p-1}z^{n-p-1} + \cdots + a_1 z + a_0$ $(p \geqslant 1)$ *be a polynomial whose zeros all lie in the closed unit disk* $|z| \leqslant 1$.

If $L_w\{Q_n(z)\} = zQ_n'(z) - w(z)\, Q_n(z)$ *where* $w(z) \in \overline{D}_{\rho,n}$ *for* $|z| \geqslant \rho^{1/(p+1)}$ $(\rho \geqslant 1)$, *then all the zeros of* $L_w\{Q_n(z)\}$ *lie in the disk* $|z| \leqslant \rho^{1/(p+1)}$.

Proof. Noticing that $a_{n-k} = 0$ $(k = 1, 2,..., p)$ yields the equalities $\sum_{m=1}^{n} z_m^k = 0$ $(k = 1, 2,..., p)$ where $z_m (m = 1, 2,..., n)$ are the zeros of Q_n, we have from [2], the notation as in the proof of the main lemma, $w^* = z_0/[z_0 - \alpha(z_0)]$ where $|\alpha(z_0)| \leqslant 1/|z_0|^p$ for $|z_0| > 1$. One notices that $\overline{B}_{|z_0|}^{(1/|z_0|^p)} = \overline{B}_{|z_0|^{p+1}}$ and arguing as in the proof of the main lemma we complete the proof.

By this lemma Theorem 1 and Theorem 2 can be generalized as follows:

THEOREM 1'. *Let* $P_n(z) = z^n + a_{n-p-1}z^{n-p-1} + \cdots + a_1 z + a_0$ $(p \geqslant 1)$ *be a polynomial whose zeros lie in the disk* $|z| \leqslant 1$.

Let $\mathscr{P}_n$ *be a family of all polynomials* $p_n(z) = a_n z^n + a_{n-p-1}z^{n-p-1} + \cdots + a_1 z + a_0$ $(p \geqslant 1, a_n \neq 0)$ *that satisfy* $|p_n(z)| \leqslant |P_n(z)|$ *on* $|z| = 1$.

Let $\rho(\rho \geqslant 1)$ *be fixed and let a and b be arbitrary complex numbers such that either* $b = 0$ *or* $a/b \in \bar{D}_{\rho,n}$.

Then for $|z| \geqslant \rho^{1/(p+1)}$

$$|bzp_n'(z) - ap_n(z)| \leqslant |bzP_n'(z) - aP_n(z)| \tag{2'}$$

for all $p_n \in \mathscr{P}_n$

THEOREM 2'. *Let* $P_n(z)$ *and* $p_n(z)$ *be polynomills as in Theorem* 1'.

If $\bar{S}$ *is the image of the region* $|z| \geqslant 1$ *by the function* $p_n(z)/P_n(z)$ *and if the numbers a and b are such that either* $b = 0$ *or* $b \neq 0$ *and* $a/b \in \bar{D}_{\rho,n}$, *then for all* z $|z| \geqslant \rho^{1/(p+1)}$ $(\rho \geqslant 1)$

$$\frac{L\{p_n(z)\}}{L\{P_n(z)\}} = \frac{ap_n(z) - bzp_n'(z)}{aP_n(z) - bzP_n'(z)} \in \bar{S}.$$

Remark 4'. Theorem 1' applied to polynolials $z^n p_m(1/z)$ and $z^n P_\ell(1/z)$ where $P_\ell(z) = a_0 + a_{p+1}z^{p+1} + \cdots + a_\ell z^\ell$ is a polynomial which does not vanish in $|z| \leqslant 1$ and $p_m(z) = b_0 + b_{p+1}z^{p+1} + \cdots + b_m z^m$ where $n \geqslant m$, $n \geqslant \ell$ yields a relation as (2)' for $|z| < \sigma^{1/(p-1)}$ $(0 < \sigma \leqslant 1)$ provided $|p_m(z)| \leqslant |P_\ell(z)|$ on $|z| = 1$ and either $b = 0$ or $a/b \in \overline{D^*_{\sigma,n}}$.

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